LECTURES ON UNCERTAINTY PRINCIPLES IN HARMONIC ANALYSIS

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1. The Heisenberg's Uncertainty Principle

Theorem 1.1. *(Heisenberg's Uncertainty Principles) Suppose f is a function in $\mathcal{S}(\mathbb{R})$ such that $\|f\|_2 = 1$. Then*

$$\left(\int_{\mathbb{R}} x^2|f(x)|^2dx\right)\left(\int_{\mathbb{R}} \xi^2|\hat{f}(\xi)|^2d\xi\right) \geq \frac{1}{16\pi^2}, \tag{1}$$

and equality holds if and only if $f(x) = Ae^{-Bx^2}$ where $B > 0$ and $|A|^2 = \sqrt{2B/\pi}$.

2. The Hardy's Uncertainty Principle

Theorem 2.1. *(The Hardy's Uncertainty Principle) Suppose f is a function in $L^1(\mathbb{R})$ such that*

$$|f(x)| \lesssim e^{-\pi a x^2} \text{ and } |\hat{f}(\xi)| \lesssim e^{-\frac{\pi}{a}\xi^2}$$

for some constant $a > 0$ and for all $x, \xi \in \mathbb{R}$. Then $f(x) = Ce^{-\pi a x^2}$ for some constnat C.

Proof. First, notice that we may assume that $a = 1$. For, let us assume that the case $a = 1$ has been proved. Take a function f satisfying the hypothesis of theorem for some $a > 0$. Define $g(x) = f(\sqrt{a}x)$ and note that g and $\hat{g}$ satisfies the hypothesis of the theorem with $a = 1$. Therefore we get that $g(x) = f(\sqrt{a}x) = Ce^{-\pi x^2}$ for all $x \in \mathbb{R}$. This implies that $f(x) = Ce^{-\pi a x^2}$. Therefore, we assume that $a = 1$.

The Gausing decay of the function allows us to extend the Fourier tranform $\hat{f}$ as an entire function to the whole complex plane. Define

$$\begin{aligned}\hat{f}(\xi + i\eta) &= \int_{\mathbb{R}} f(x)e^{-2\pi i x(\xi+i\eta)}dx \\ &= \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}e^{2\pi x\eta}dx\end{aligned}$$

Using the estimate on decay of f we get

$$\begin{aligned}|\hat{f}(\xi + i\eta)| &\leq \int_{\mathbb{R}} e^{-\pi x^2}e^{2\pi x\eta}dx \\ &= \int_{\mathbb{R}} e^{-\pi(x-\eta)^2}e^{\pi\eta^2}dx \\ &\leq e^{\pi\eta^2} \text{ for all } \xi, \eta \in \mathbb{R}.\end{aligned}$$

Therefore, thefunction $F(z) = e^{\pi z^2}\hat{f}(z)$, $z = \xi + i\eta \in \mathbb{C}$, is entire and on imaginary axis ($\xi = 0$), we get

$$|F(i\eta)| \leq |e^{\pi(i\eta)^2}\hat{f}(z)| \leq 1.$$

Also, by the hypothesis on $\hat{f}$ we know that on the real axis ($\eta = 0$) also, the function F is bounded by 1.

The Phragmen-Lindelöf principle for sectorial domains (as stated below) may be applied (with some extra work) to this situation and we can deduce that the function F is bounded

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on the entire Complex plane. Then by Liouville's theorem we see that F is constant function. This completes the proof of the theorem. □

2.1. Maximum modulus principle.

Theorem 2.2. *(Maximum modulus principle) Let U be a connected open set of $\mathbb{C}$ and $f : U \to \mathbb{C}$ be a holomorphic function on U. If zz_0 is a point in U such that*

$$|f(z) \leq |f(z_0)|$$

for all z in a neighbourhood $N_{z_0} \subset U$, then f is a constant function on U.

In other words, the maximum modulus principle says that if f is a non-constant holomorphic function on U then $|f|$ cannot attain its local maximum at a point in U. Further, if U is a bounded domain then the maximum modulus principle asserts that $|f|$ attains its maximum on the the boundary of the domain U. However, this phenomenon is no longer true for the unbounded domains. For example, the function $f(z) := e^{-z^2}$ is holomorphic in the upper half-plane. On the boundary, i.e., on real axis, the function is bounded by 1. However, for points $z = iy$ on the imaginary axis we see that $f(iy) := e^{y^2}$, which is unbounded.

There are many variants of the maximum modulus principle for particular domains (bounded or unbounded) with an extra assumption on the growth of the function for unbounded domains. In particular, the theorems known as Hadamard's three-lines (or three circles) principle and Phragmen-Lindelöf principle are the most important.

Theorem 2.3. *(Hadamard's three-line principle) Let $S := \{x + iy : a < x < b\}$ denotes a strip in $\mathbb{C}$ for some real numbers $a < b$. Let $f : \bar{S} \to \mathbb{C}$ be a bounded function such that it is holomorphic on S and continuous on $\bar{S}$. If*

$$M(x) := \sup_y |f(x + iy)|.$$

Then $\log M(x)$ is a convex function of $[a, b]$.
In other words, $x \in [a, b]$ is written as the convex hull $x = ta + (1 - t)b,\ 0 \leq t \leq 1$, then

$$M(x) \leq M(a)^t M(b)^{1-t}.$$

Theorem 2.4. *(The Phragmen-Lindelöf principle for the half-plane) Let f be holomorphic on the upper half-plane $\mathbb{H}$ and continuous on the boundary $\partial\mathbb{H}$. Let*

$$M(r) := \max\{|f(z)| : z \in \mathbb{H}, |z| = r\}.$$

If $|f(x)| \leq 1$ for all $x \in \mathbb{R}$ and $\frac{1}{r}\log M(r) \to 0$ as $r \to \infty$, then $|f(z)| \leq 1$ for all $z \in \mathbb{H}$.

As a consequence of the above result, one can obtain the following Phragmen-Lindelöf principle for the sectorial domains in the complex plane. This version of the Phragmen-Lindelöf principle may be used to prove the Hardy's uncertainty principle.

Theorem 2.5. *(The Phragmen-Lindelöf principle for sectorial domains) Let $\alpha > \frac{1}{2}$ and let $U_\alpha := \{z = re^{i\theta} : |\theta| < \frac{\pi}{2\alpha}\}$. Suppose f is a holomorphic function on U_α and continuous of the boundary ∂U_α. If $|f(z)| \leq 1$ for all $z \in \partial U_\alpha$ and $|f(z)| \leq Ce^{|z|^\beta}$ for all $z \in U_\alpha$ for some constants $\beta < \alpha$ and $C > 0$, then $|f(z)| \leq 1$ for all $z \in U_\alpha$.*

3. The Benedick's inequality

Theorem 3.1. *(The Benedick's inequality) Suppose f is a non-zero function in $L^1(\mathbb{R})$. Denote $\Sigma(f) = \{x \in \mathbb{R} : f(x) \neq 0\}$. Then*

$$|\Sigma(f)||\Sigma(\hat{f})| = \infty.$$

Proof. Proof is by contradiction. Suppose for some non-zero function $f \in L^1(\mathbb{R})$

$$|\Sigma(f)||\Sigma(\hat{f})| < \infty.$$

Observe that, we may assume that $|\Sigma(f)| < 1$. For, a given function f with $|\Sigma(f)| = K$, we can consider the function $f_K(x) := f(\frac{x}{2K})$.

Note that $\Sigma(f_K) = \{x \in \mathbb{R} : f(x/2K) \neq 0\}$ and hence $\frac{|\Sigma(f)|}{2K} = |\Sigma(f_K)|$. Then the known result applied to f_K gives us the desired result for the original function f.

Consider

$$\int_{[0,1]} \sum_{n\in\mathbb{Z}} \chi_{\Sigma(\hat{f})}(\xi + n) d\xi = \int_{\mathbb{R}} \chi_{\Sigma(\hat{f})}(\xi) d\xi = |\Sigma(\hat{f})| < \infty,$$

and similarly

$$\int_{[0,1]} \sum_{n\in\mathbb{Z}} \chi_{\Sigma(f)}(x + n) dx = \int_{\mathbb{R}} \chi_{\Sigma(f)}(x) dx = |\Sigma(f)| < 1.$$

From the above expressions we conclude that

- $\sum_{n\in\mathbb{Z}} \chi_{\Sigma(\hat{f})}(\xi + n) < \infty$ for almost every $\xi \in [0, 1]$. In other words, there is a set $E \subset [0, 1]$ with $|E| = 1$ such that
$$\sum_{n\in\mathbb{Z}} \chi_{\Sigma(\hat{f})}(\xi + n) < \infty \text{ for every } \xi \in E.$$
Consequently, for every $\xi \in E$, $\hat{f}(\xi + n) \neq 0$ for only finitely many n.
- There is a set $F \subset [0, 1]$ of positive measure such that for all $x \in F$ we have
$$\sum_{n\in\mathbb{Z}} \chi_{\Sigma(f)}(x + n) < 1.$$
This gives us that $f(x + n) = 0$ for all $x \in F$ and all $n \in \mathbb{Z}$.

Now for $y \in E$ consider the function $F_y := \sum_{n\in\mathbb{Z}} f(x+n)e^{-2\pi iy(x+n)}$. Observe that $F_y \in L^1([0, 1])$ as $f(x)e^{-2\pi iyx}$ is in $L^1(\mathbb{R})$.

Further, since for $y \in E$ we have that $\hat{f}(y + n) \neq 0$ for only finitely many n, we see that the Fourier series of F_y at x is given by $F_y(x) = \sum_{n\in\mathbb{Z}} \hat{f}(y+n)e^{-2\pi iyx}$. Since this a finite sum, we can extend the function F_y as an entire function to the whole complex plane $\mathbb{C}$ by

$$F_y(z) = \sum_{n\in\mathbb{Z}} \hat{f}(y + n)e^{-2\pi iyz}, \ z \in \mathbb{C}.$$

Now we use the other definition of F_y and see that for $x \in F$ we have that

$$|F_y(x)| \leq \sum_{n\in\mathbb{Z}} |f(x + n)| = 0.$$

The set f has positive measure and F_y is entire if $y \in E$, then we can conclude that F_y is identically zero function for $y \in E$. This further implies that $\hat{f}(y+n) = 0$ for all $y \in E$ and $n \in \mathbb{Z}$. Since $E \subset [0, 1]$ has full measure we see that $\hat{f}$ is identically zero function. This completes the proof.

□

References

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